Sigma Functions for Telescopic Curves*

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Abstract

In this paper, we consider a symplectic basis of the first cohomology group and the sigma functions for algebraic curves expressed by a canonical form using a finite sequence $(a_1, \cdots, a_t)$ of positive integers whose greatest common divisor is equal to one (Miura [13]). The idea is to express a non-singular algebraic curve by affine equations of t variables whose orders at infinity are $(a_1, \cdots, a_t)$. We construct a symplectic basis of the first cohomology group and the sigma functions for telescopic curves, i.e., the curves such that the number of defining equations is exactly $t-1$ in the Miura canonical form. The largest class of curves for which such construction has been obtained thus far is (n, s)-curves ([3][15]), which are telescopic because they are expressed in the Miura canonical form with $t = 2$, $a_1 = n$, and $a_2 = s$, and the number of defining equations is one.

1 Introduction

In recent years, the theory of Abelian functions is attracting increasing interest in mathematical physics and applied mathematics. In particular, the sigma functions associated with algebraic curves with higher genus have been studied actively. In this paper, we construct the sigma functions for a class of algebraic curves for which such construction has not been obtained thus far.

Let C be a compact Riemann surface with genus g, and $H^1(C, \mathbb{C})$ the first cohomology group, which is defined by the linear space of second kind differentials modulo meromorphic exact forms. We say a meromorphic differential on C to be second kind if it is locally exact.

We consider a basis of $H^1(C, \mathbb{C})$ consisting of $\dim_{\mathbb{C}} H^1(C, \mathbb{C}) = 2g$ elements (cf. [11], pp.29-31, Theorem 8.1,8.2) . In particular, in order to construct the sigma functions, we wish to construct a basis (symplectic basis) $\{du_i, dr_i\}_{i=1}^{g}$ of $H^1(C, \mathbb{C})$ such that

1. du_i is holomorphic on C for each i, and

2. $du_i \circ du_j = dr_i \circ dr_j = 0$ and $du_i \circ dr_j = \delta_{ij}$ for each i, j,

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where the operator $\circ$ is the intersection form on $H^1(C,\mathbb{C})$ defined by

$$\eta \circ \eta' = \sum_p \mathrm{Res}(\int^p \eta)\eta'(p)$$

for second kind differentials η, η' (the summation is over the singular points of η and η', and Res denotes the residue at the point).

In order to express defining equations of C, we use a canonical form for expressing non-singular algebraic curves proposed by Miura [13]. Given a finite sequence $(a_1,\ldots,a_t)$ of positive integers whose greatest common divisor is equal to one, Miura [13] proposed a non-singular algebraic curve determined by the sequence $(a_1,\ldots,a_t)$, which is called $(a_1,\ldots,a_t)$-curve. The idea is to express a non-singular algebraic curve by affine equations of t variables whose orders at infinity are $(a_1,\cdots,a_t)$. Any non-singular algebraic curve is birationally equivalent to some $(a_1,\ldots,a_t)$-curve (cf. [13]).

Klein [9][10] constructed a symplectic basis of the first cohomology group and the sigma functions for the hyperelliptic curves with genus g, which are expressed in the Miura canonical form with $t=2$, $a_1=2$, and $a_2=2g+1$. Buchstaber et al. [3] and Nakayashiki [15] constructed them for more general plane algebraic curves called (n,s)-curves, which are expressed in the Miura canonical form with $t=2$, $a_1=n$, and $a_2=s$. In this paper, we construct them for the telescopic $(a_1,\ldots,a_t)$-curves, i.e., the $(a_1,\ldots,a_t)$-curves such that for any i $(2\le i\le t)$

$$\frac{a_i}{d_i} \in \frac{a_1}{d_{i-1}}\mathbb{N} + \cdots + \frac{a_{i-1}}{d_{i-1}}\mathbb{N}, \quad d_i := \mathrm{GCD}\{a_1,\cdots,a_i\}.$$

The class of the telescopic $(a_1,\ldots,a_t)$-curves is equal to that of the $(a_1,\ldots,a_t)$-curves such that the number of the defining equations in the Miura canonical form is exactly $t-1$ (cf. [18]). Since the (n,s)-curves are telescopic, our curves contain them strictly. Recently, Matsutani [12] constructed them for the $(3,4,5)$-curves, which are not telescopic.

The plan of this paper is as follows. In section 2, we define the $(a_1,\ldots,a_t)$-curves. Because there exists no English paper for the $(a_1,\ldots,a_t)$-curves, we give complete proofs in Appendix. In section 3, we construct holomorphic 1-forms $\{du_i\}_{i=1}^g$ for the telescopic $(a_1,\ldots,a_t)$-curves. In section 4, we construct second kind differentials $\{dr_i\}_{i=1}^g$ for the telescopic $(a_1,\ldots,a_t)$-curves, and show that the set $\{du_i,dr_i\}_{i=1}^g$ is a symplectic basis of the first cohomology group. In section 5, we construct the sigma functions associated with the telescopic $(a_1,\ldots,a_t)$-curves.

Throughout this paper, $\mathbb{N},\mathbb{N}_+,\mathbb{Z}$, and $\mathbb{C}$ denote the non-negative integers, the positive integers, the integers, and the complex numbers, respectively.

2 $(a_1,\ldots,a_t)$-Curves

Miura [13] proposed a canonical form for expressing non-singular algebraic curves. Let $t\ge 2$, and $a_1,\ldots,a_t$ positive integers such that $\mathrm{GCD}\{a_1,\ldots,a_t\}=$

1. We denote $A_t := (a_1, \ldots, a_t) \in \mathbb{N}_+^t$ and $\langle A_t \rangle := a_1\mathbb{N} + \cdots + a_t\mathbb{N}$, assuming that the order of $a_1, \ldots, a_t$ is fixed. For the map $\Psi : \mathbb{N}^t \to \langle A_t \rangle$ defined by $\Psi((m_1, \ldots, m_t)) = \sum_{i=1}^t a_i m_i$, we define the order $<$ in $\mathbb{N}^t$ so that $M < M'$ for $M = (m_1, \ldots, m_t)$ and $M' = (m'_1, \ldots, m'_t)$ if

1. $\Psi(M) < \Psi(M')$ or
2. $\Psi(M) = \Psi(M')$ and $m_1 = m'_1, \ldots, m_{i-1} = m'_{i-1}, m_i > m'_i$ for some i $(1 \le i \le t)$.

Let $M(a)$ be the minimum element with respect to the order $<$ in $\mathbb{N}^t$ satisfying $\Psi(M) = a \in \langle A_t \rangle$. We define $B(A_t) \subseteq \mathbb{N}^t$ and $V(A_t) \subseteq \mathbb{N}^t \backslash B(A_t)$ by

$$B(A_t) = \{M(a) \mid a \in \langle A_t \rangle\}$$

and

$$V(A_t) = \{L \in \mathbb{N}^t \backslash B(A_t) \mid L = M + N, M \in \mathbb{N}^t \backslash B(A_t), N \in \mathbb{N}^t \Rightarrow N = (0, \ldots, 0)\},$$

respectively.

Hereafter, $\mathbb{C}[X] := \mathbb{C}[X_1, \ldots, X_t]$ denotes the polynomial ring over $\mathbb{C}$ of t-variables $X_1, \ldots, X_t$. For $A \subset \mathbb{C}[X]$, $\mathrm{Span}\{A\}$ and (A) denote the linear space over $\mathbb{C}$ generated by A and the ideal in $\mathbb{C}[X]$ generated by A, respectively. Also, $X^M, M = (m_1, \ldots, m_t)$, denotes $X^M = X_1^{m_1} \cdots X_t^{m_t}$ for simplicity.

For $M \in V(A_t)$, we define the polynomial $F_M(X) \in \mathbb{C}[X]$ by

$$F_M(X) = X^M - X^L - \sum_{\{N \in B(A_t) \mid \Psi(N) < \Psi(M)\}} \lambda_N X^N, \qquad \lambda_N \in \mathbb{C}, \qquad (1)$$

where L is the element of $B(A_t)$ such that $\Psi(L) = \Psi(M)$. We assume that the set of the polynomials $\{F_M \mid M \in V(A_t)\}$ satisfies the following condition:

$$\mathrm{Span}\{X^N \mid N \in B(A_t)\} \cap (\{F_M \mid M \in V(A_t)\}) = \{0\}. \qquad (2)$$

Let $I := (\{F_M \mid M \in V(A_t)\})$, $R := \mathbb{C}[X]/I$, x_i the image of X_i for the projection $\mathbb{C}[X] \to R$, and K the total quotient ring of R. Then, we have the following three propositions. For complete proofs, see Appendix.

Proposition 2.1 (Miura [13]).

(i) The set $\{x^N \mid N \in B(A_t)\}$ is a basis of R over $\mathbb{C}$, where $x := (x_1, \ldots, x_t)$.

(ii) The ring R is an integral domain, hence, K is the quotient field of R.

(iii) The field K is an algebraic function field of one variable over $\mathbb{C}$.

(iv) There exists a discrete valuation v_∞ of K such that $(x_i)_\infty = a_i v_\infty$ for any i, where $(x_i)_\infty$ denotes the pole divisor of x_i (cf.[19] p.19).

Let $C^{\mathrm{aff}} := \{(z_1, \ldots, z_t) \in \mathbb{C}^t \mid f(z_1, \ldots, z_t) = 0, \ \forall f \in I\}$. From Proposition 2.1 (ii) (iii), C^{aff} is an affine algebraic curve in $\mathbb{C}^t$. Hereafter, we assume that C^{aff}

is non-singular. For $k \in \mathbb{N}$, we define $L(kv_\infty) := \{f \in K \mid (f) + kv_\infty \geq 0\} \cup \{0\}$, where (f) denotes the divisor of f, i.e., $(f) = \sum_v v(f) \cdot v$.

Proposition 2.2 (Miura [13]). *(i)* $R = \bigcup_{k=0}^{\infty} L(kv_\infty)$.

(ii) The map ϕ

$$C^{aff} \to \{\textit{discrete valuation of } K\} \backslash \{v_\infty\}$$

$$p \to v_p$$

is bijective, where v_p *is the discrete valuation corresponding to* $p \in C^{aff}$ *(cf. [17], p.21,22).*

Let C be the compact Riemann surface corresponding to C^{aff}. From Proposition 2.2 (ii), we can obtain C by adding one point ∞ to C^{aff}, where the discrete valuation corresponding to ∞ is v_∞. It is known that any non-singular algebraic curve is birationally equivalent to such C for some A_t (cf. [13]). Hereafter, we represent each curve C by the sequence $A_t = (a_1, \ldots, a_t)$ and call $(a_1, \ldots, a_t)$-curve for short.

The sequence $A_t = (a_1, \ldots, a_t)$ is called telescopic if for any i $(2 \leq i \leq t)$

$$\frac{a_i}{d_i} \in \frac{a_1}{d_{i-1}}\mathbb{N} + \cdots + \frac{a_{i-1}}{d_{i-1}}\mathbb{N}, \quad d_i := \mathrm{GCD}\{a_1, \cdots, a_i\}.$$

Note that $A_2 = (a_1, a_2)$ is always telescopic.

Proposition 2.3 (Miura [13]). *If* A_t *is telescopic, the condition (2) is met, and we have the followings.*

(i) $B(A_t) = \{(m_1, \ldots, m_t) \in \mathbb{N}^t \mid 0 \leq m_i \leq d_{i-1}/d_i - 1,\ 2 \leq i \leq t\}$.

(ii) $V(A_t) = \{(0, \ldots, 0, d_{i-1}/d_i, 0, \ldots, 0) \mid 2 \leq i \leq t\}$, *where* d_{i-1}/d_i *is in the* i*-th component.*

(iii) The genus g *of* C *is*

$$g = \frac{1}{2}\left\{(1 - a_1) + \sum_{i=2}^{t}\left(\frac{d_{i-1}}{d_i} - 1\right)a_i\right\}. \tag{3}$$

Note that $\sharp V(A_t)$ is the number of the defining equations, where $\sharp$ denotes the number of elements. From Lemma C.1 (iv) in Appendix, we obtain $\sharp V(A_t) \geq t-1$. If A_t is telescopic, then from Proposition 2.3 (ii) we obtain $\sharp V(A_t) = t-1$. On the other hand, Suzuki [18] proved that if $\sharp V(A_t) = t - 1$, then A_t is telescopic.

If $A_t = (a_1, \ldots, a_t)$ is telescopic, we call the $(a_1, \ldots, a_t)$-curve telescopic $(a_1, \ldots, a_t)$-curve. From Proposition 2.3, the defining equations of a telescopic $(a_1, \ldots, a_t)$-curve is given as follows: for $2 \leq i \leq t$,

$$F_i(X_1, \ldots, X_t) = X_i^{d_{i-1}/d_i} - \prod_{j=1}^{t} X_j^{m_{ij}} - \sum_{j_1, \ldots, j_t} \lambda_{j_1, \ldots, j_t}^{(i)} X_1^{j_1} \cdots X_t^{j_t},$$

where $(m_{i1},\dots,m_{it}) \in B(A_t)$, $\sum_{j=1}^t a_j m_{ij} = a_i d_{i-1}/d_i$, $\lambda^{(i)}_{j_1,\dots,j_t} \in \mathbb{C}$, and $(j_1,\dots,j_t)$ runs over $(j_1,\dots,j_t) \in B(A_t)$ with $\sum_{k=1}^t a_k j_k < a_i d_{i-1}/d_i$.

EXAMPLE 1. $A_2 = (n,s)$ for relatively prime positive integers n and s.

Since $A_2 = (n,s)$ is telescopic, from Proposition 2.3 (ii), we have $V(A_2) = \{(0,n)\}$. Hence, the (n,s)-curves (cf. [15], p.182) are given by the following defining equation:

$$F_2(X_1,X_2) = X_2^n - X_1^s - \sum_{nj_1+sj_2<ns} \lambda^{(2)}_{j_1,j_2} X_1^{j_1} X_2^{j_2}.$$

In particular, we obtain the elliptic curves if $n=2$ and $s=3$, and the hyperelliptic curves with genus g if $n=2$ and $s=2g+1$.

EXAMPLE 2. $A_3 = (4,6,5)$.

Since $A_3 = (4,6,5)$ is telescopic, from Proposition 2.3 (ii), we have $V(A_3) = \{(0,2,0),(0,0,2)\}$. Hence, the $(4,6,5)$-curves are given by the following defining equations:

$$F_2(X_1,X_2,X_3) = X_2^2 - X_1^3 - \lambda^{(2)}_{0,1,1}X_2X_3 - \lambda^{(2)}_{1,1,0}X_1X_2 - \lambda^{(2)}_{1,0,1}X_1X_3 - \lambda^{(2)}_{2,0,0}X_1^2$$
$$-\lambda^{(2)}_{0,1,0}X_2 - \lambda^{(2)}_{0,0,1}X_3 - \lambda^{(2)}_{1,0,0}X_1 - \lambda^{(2)}_{0,0,0}$$

and

$$F_3(X_1,X_2,X_3) = X_3^2 - X_1X_2 - \lambda^{(3)}_{1,0,1}X_1X_3 - \lambda^{(3)}_{2,0,0}X_1^2 - \lambda^{(3)}_{0,1,0}X_2 - \lambda^{(3)}_{0,0,1}X_3$$
$$-\lambda^{(3)}_{1,0,0}X_1 - \lambda^{(3)}_{0,0,0}.$$

3 Holomorphic 1-Forms for the Telescopic Curves

Let C be a telescopic $(a_1,\dots,a_t)$-curve, and $\Gamma(C,\Omega^1_C)$ the linear space consisting of holomorphic 1-forms on C. In this section, we construct a basis of $\Gamma(C,\Omega^1_C)$. Let G be the matrix defined by

$$G := \begin{pmatrix} \dfrac{\partial F_2}{\partial X_1} & \cdots & \dfrac{\partial F_2}{\partial X_t} \\ \cdots & \cdots & \cdots \\ \dfrac{\partial F_t}{\partial X_1} & \cdots & \dfrac{\partial F_t}{\partial X_t} \end{pmatrix},$$

and G_i the matrix obtained by removing the i-th column from G. Then, we have the following theorem.

Theorem 3.1. *The set*

$$P := \left\{ \frac{x_1^{k_1}\cdots x_t^{k_t}}{\det G_1(x)} dx_1 \mid (k_1,\dots,k_t) \in B(A_t),\ 0 \le \sum_{i=1}^t a_i k_i \le 2g-2 \right\}$$

is a basis of $\Gamma(C,\Omega^1_C)$ *over* $\mathbb{C}$*, where* $\det G_1(x)$ *denotes* $\det G_1(X=x)$*.*

We rearrange the elements of P in the ascending order with respect to the order at ∞, and write $\{du_1,\ldots,du_g\}$.

In order to prove Theorem 3.1, we need some lemmas.

Lemma 3.1. *If* $\det G_i(p)\neq 0$ *for* $p=(p_1,\ldots,p_t)\in C^{aff}$ *and* $1\le i\le t$*, then* $v_p(x_i-p_i)=1$*.*

Proof. Without loss of generality, we assume $i=1$. Suppose $v_p(x_1-p_1)\ge 2$. Then, there exists k $(2\le k\le t)$ such that $v_p(x_k-p_k)=1$. In fact, if $v_p(x_k-p_k)\ge 2$ for any k, then $v_p(f)\ge 2$ or $v_p(f)=0$ for any $f\in R$. Then, $v_p(g)\ge 2$ or $v_p(g)=0$ for any $g\in R_p$, where R_p be the localization of R at p. This contradicts that R_p is a discrete valuation ring.

There exist $\{\gamma_{ij},\delta^{(i)}_{j_1,\ldots,j_t}\}\in\mathbb{C}$ such that for $2\le i\le t$

$$F_i(X_1,\ldots,X_t)=\sum_{j=1}^{t}\gamma_{ij}(X_j-p_j)+\sum_{j_1+\cdots+j_t\ge 2}\delta^{(i)}_{j_1,\ldots,j_t}(X_1-p_1)^{j_1}\cdots(X_t-p_t)^{j_t},$$

where $\gamma_{ij}=\frac{\partial F_i}{\partial X_j}(p)$. Since $F_i(x_1,\ldots,x_t)=0$ and $v_p(x_1-p_1)\ge 2$, we have $v_p\left(\sum_{j=2}^{t}\gamma_{ij}(x_j-p_j)\right)=v_p\left((x_k-p_k)(\sum_{j=2}^{t}\gamma_{ij}\frac{x_j-p_j}{x_k-p_k})\right)\ge 2$. Since $v_p(x_k-p_k)=1$, we have $\sum_{j=2}^{t}\gamma_{ij}b_j=0$, where $b_j=\left(\frac{x_j-p_j}{x_k-p_k}\right)(p)$. Hence, we obtain

$$G_1(p)\begin{pmatrix}b_2\\ \cdot\\ \cdot\\ b_t\end{pmatrix}=\begin{pmatrix}0\\ \cdot\\ \cdot\\ 0\end{pmatrix}.$$

Since $b_k=1(\neq 0)$, we have $\det G_1(p)=0$. This contradicts the assumption of Lemma 3.1. Hence, we obtain $v_p(x_1-p_1)=1$.

□

Lemma 3.2. *(i) As an element of* K*, we have* $\det G_1(x)\neq 0$*.*

(ii) $div\left(\dfrac{dx_1}{\det G_1(x)}\right)=(2g-2)\infty$*.*

Proof. Since the differential $d\left(F_i(x_1,\ldots,x_t)\right)=0$ for any i, we have

$$G(x)\begin{pmatrix}dx_1\\ \cdot\\ \cdot\\ dx_t\end{pmatrix}=\begin{pmatrix}0\\ \cdot\\ \cdot\\ 0\end{pmatrix}\quad .$$

By multiplying some elementary matrices on the left, the above equation be-

comes

$$\begin{pmatrix} w_2 & z_{22} & z_{23} \cdots z_{2t} \\ w_3 & 0 & z_{33} \cdots z_{3t} \\ & & \cdots & \\ w_t & 0 & \cdots & z_{tt} \end{pmatrix} \begin{pmatrix} dx_1 \\ \cdot \\ \cdot \\ dx_t \end{pmatrix} = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix}.$$

Since C^{aff} is non-singular for any $p \in C^{\mathrm{aff}}$, there exists i such that $\det G_i(p) \neq 0$. Hence, we have $w_t \neq 0$ or $z_{tt} \neq 0$ as elements of K. Since $v_\infty(x_j) = -a_j$, we have $x_j \notin \mathbb{C}$, hence, $dx_j \neq 0$ for any j. Since $w_t dx_1 = z_{tt} dx_t$, we have $w_t \neq 0$ and $z_{tt} \neq 0$. Hence, by multiplying some elementary matrices on the left, the above equation becomes

$$\begin{pmatrix} w_2' & z_{22} & z_{23} \cdots 0 \\ w_3' & 0 & z_{33} \cdots 0 \\ & & \cdots & \\ w_t & 0 & \cdot\cdot & z_{tt} \end{pmatrix} \begin{pmatrix} dx_1 \\ \cdot \\ \cdot \\ dx_t \end{pmatrix} = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix}.$$

Similarly, we obtain

$$\begin{pmatrix} w_2'' & z_{22} & 0 \cdots 0 \\ w_3'' & 0 & z_{33} \cdots 0 \\ & & \cdots & \\ w_t'' & 0 & \cdots & z_{tt} \end{pmatrix} \begin{pmatrix} dx_1 \\ \cdot \\ \cdot \\ dx_t \end{pmatrix} = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix},$$

where $w_2'', \ldots, w_t'', z_{22}, \ldots, z_{tt} \in K$ are non-zero. Hence, we obtain $\det G_1(x) = \pm z_{22} \cdots z_{tt} \neq 0$, which complete the proof of (i).

Next, we prove that the 1-form $dx_1/\det G_1(x)$ is both holomorphic and non-vanishing on C^{aff}. When $\det G_1(p) \neq 0$ for $p \in C^{\mathrm{aff}}$, from Lemma 3.1, $dx_1/\det G_1(x)$ is both holomorphic and non-vanishing at p. Suppose $\det G_1(p) = 0$ for $p \in C^{\mathrm{aff}}$. Since C^{aff} is non-singular at p, there exists i $(2 \leq i \leq t)$ such that $\det G_i(p) \neq 0$. Since $w_i'' dx_1 + z_{ii} dx_i = 0$, we have $w_i'' z_{22} \cdots \widehat{z_{ii}} \cdots z_{tt} dx_1 + z_{22} \cdots z_{tt} dx_i = 0$, i.e., $\det G_i(x) dx_1 \pm \det G_1(x) dx_i = 0$, where $\widehat{z_{ii}}$ denotes to remove z_{ii}. Since $\det G_1(x) \neq 0$ and $\det G_i(x) \neq 0$, we have $dx_1/\det G_1(x) = \pm dx_i/\det G_i(x)$. Hence, from $\det G_i(p) \neq 0$ and Lemma 3.1, $dx_1/\det G_1(x)$ is holomorphic and non-vanishing at p. On the other hand, by Riemann-Roch's theorem, we have $\deg \operatorname{div}(dx_1/\det G_1(x)) = 2g - 2$, which complete the proof of (ii).

□

Proof of Theorem 3.1. From Lemma 3.2 and Proposition 2.1 (i), we have $P \subset \Gamma(C, \Omega_C^1)$, and the elements of P are linearly independent. Since $\dim_{\mathbb{C}} \Gamma(C, \Omega_C^1) = g$, it is sufficient to prove $\sharp P = g$. It is well-known that there are g gap values at ∞ from 0 to $2g - 1$. Since $\dim_{\mathbb{C}} L((2g-1)v_\infty) = \dim_{\mathbb{C}} L((2g-2)v_\infty) = g$ (Riemann-Roch's theorem), $2g-1$ is a gap value at ∞. Hence, from Proposition 2.1 (i) and Proposition 2.2 (i), we have $\sharp\{(k_1, \ldots, k_t) \in B(A_t) \mid 0 \leq \sum_{i=1}^t a_i k_i \leq 2g - 2\} = g$, which complete the proof of Theorem 3.1.

□

4 Second Kind Differentials for the Telescopic Curves

In this section, we construct dr_i for a telescopic $(a_1, \dots, a_t)$-curve C. For $2 \le i \le t$ and $1 \le j \le t$, we divide $F_i(Y_1, \dots, Y_{j-1}, X_j, \dots, X_t)$ by $X_j - Y_j$, where we regard X_j as a variable. Let $h_{ij} \in \mathbb{C}[X_1, \dots, X_t, Y_1, \dots, Y_t]$ be the quotient, i.e.,

$$F_i(Y_1, \dots, Y_{j-1}, X_j, \dots, X_t) = h_{ij} \cdot (X_j - Y_j) + F_i(Y_1, \dots, Y_j, X_{j+1}, \dots, X_t), \quad (4)$$

and H the matrix defined by

$$H := \begin{pmatrix} h_{22} & \dots & h_{2t} \\ \dots & \dots & \dots \\ h_{t2} & \dots & h_{tt} \end{pmatrix}.$$

We consider the 1-form

$$\Omega(x, y) := \frac{\det H(x, y)}{(x_1 - y_1) \det G_1(x)} dx_1$$

and the bilinear form (cf. [15], p.181, 2.4)

$$\hat{\omega}(x, y) := d_y \Omega(x, y) + \sum c_{i_1, \dots, i_t, j_1, \dots, j_t} \frac{x_1^{i_1} \cdots x_t^{i_t} y_1^{j_1} \cdots y_t^{j_t}}{\det G_1(x) \det G_1(y)} dx_1 dy_1 \quad (5)$$

on $C \times C$, where $x = (x_1, \dots, x_t)$, $y = (y_1, \dots, y_t)$, $c_{i_1, \dots, i_t, j_1, \dots, j_t} \in \mathbb{C}$, $(i_i \dots, i_t) \in B(A_t)$ with $0 \le \sum_{k=1}^t a_k i_k \le 2g - 2$, and $(j_i \dots, j_t) \in B(A_t)$.

We take a basis $\{\alpha_i, \beta_i\}_{i=1}^g$ of the homology group $H_1(C, \mathbb{Z})$ such that their intersection numbers are $\alpha_i \circ \alpha_j = \beta_i \circ \beta_j = 0$ and $\alpha_i \circ \beta_j = \delta_{ij}$.

DEFINITION 4.1. (cf. [15], p.181, 2.4) Let $\Delta := \{(p, p) \mid p \in C\}$. A meromorphic symmetric bilinear form $\omega(x, y)$ on $C \times C$ is called a normalized fundamental form if the following conditions are satisfied.

(i) $\omega(x, y)$ is holomorphic except Δ where it has a double pole. For $p \in C$, we take a local coordinate s around p. Then, the expansion in $s(x)$ at $s(y)$ is of the form

$$\omega(x, y) = \left(\frac{1}{(s(x) - s(y))^2} + regular \right) ds(x) ds(y).$$

(ii) $\displaystyle\int_{\alpha_i} \omega = 0$ for any i, where the integration is with respect to any one of the variables.

Normalized fundamental form exists and unique (cf. [15] p.182). Then, we have the following theorem.

Theorem 4.1.

(i) There exists a set of $c_{i_1,\dots,i_t,j_1,\dots,j_t}$ such that $\hat{\omega}(x,y)=\hat{\omega}(y,x)$.

We take $c_{i_1,\dots,i_t,j_1,\dots,j_t}$ such that $\hat{\omega}(x,y)=\hat{\omega}(y,x)$. Then, we have the followings.

(ii) The bilinear form $\hat{\omega}$ satisfies the condition (i) of Definition 4.1.

(iii) For $du_i := (x_1^{k_{i,1}}\cdots x_t^{k_{i,t}}/\det G_1(x))dx_1$, we define

$$dr_i := \sum_{j_1,\dots,j_t} c_{k_{i,1},\dots,k_{i,t},j_1,\dots,j_t}\frac{y_1^{j_1}\cdots y_t^{j_t}}{\det G_1(y)}dy_1.$$

Then, dr_i is a second kind differential for any i, and the set $\{du_i, dr_i\}_{i=1}^{g}$ is a symplectic basis of $H^1(C,\mathbb{C})$.

In order to prove Theorem 4.1, we generalize Lemma 2,3,4,5,6 in [15] to the telescopic $(a_1,\dots,a_t)$-curves.

Let B be the set of branch points for the map $x_1 : C \to \mathbb{P}^1, (x_1,\dots,x_t) \to [x_1 : 1]$ (cf. [17], p.24, Example 2.2). Since the ramification index of the map x_1 at ∞ is a_1, we have $\deg x_1 = a_1$ (cf. [17], p.28, Proposition 2.6). For $p\in C$, we set $x_1^{-1}(x_1(p)) = \{p^{(0)}, p^{(1)},\dots,p^{(a_1-1)}\}$ with $p=p^{(0)}$, where the same $p^{(i)}$ is listed according to its ramification index.

Lemma 4.1. *Let U be a domain in $\mathbb{C}$, $f(z_1,z_2)$ a holomorphic function on $U\times U$, and $g(z) := f(z,z)$. If $g\equiv 0$ on U, then there exists a holomorphic function $h(z_1,z_2)$ on $U\times U$ such that $f(z_1,z_2)=(z_1-z_2)h(z_1,z_2)$.*

Proof. Let $h(z_1,z_2) := f(z_1,z_2)/(z_1-z_2)$. Given z_1, $h(z_1,\cdot\,)$ has a singularity only at z_1, where its singularity is removable, i.e., $h(z_1,\cdot\,)$ is holomorphic on U. Similarly, $h(\,\cdot,z_2)$ is holomorphic on U. Hence, h is holomorphic on $U\times U$. □

Lemma 4.2. *The 1-form $\Omega(x,y)$ is holomorphic except $\Delta\cup\{(p^{(i)},p)\mid i\neq 0,\ p\in B \text{ or } p^{(i)}\in B\}\cup C\times\{\infty\}\cup\{\infty\}\times C$.*

Proof. Since $dx_1/\det G_1(x)$ is holomorphic on C (cf. Lemma 3.2), $\Omega(x,y)$ is holomorphic except $\Delta\cup\{(p^{(i)},p)\mid p\in C, i\neq 0\}\cup C\times\{\infty\}\cup\{\infty\}\times C$. We prove that $\Omega(x,y)$ is holomorphic on $\{(p^{(i)},p)\mid i\neq 0, p\notin B, p^{(i)}\notin B\}$. From (4), we obtain

$$F_i(X_1,\dots,X_t)=\sum_{j=1}^{t} h_{ij}\cdot(X_j-Y_j)+F_i(Y_1,\dots,Y_t). \tag{6}$$

Set $X=x$ and $Y=y$, then we have

$$\sum_{j=1}^{t} h_{ij}(x,y)\cdot(x_j-y_j)=0.$$

Take $(p^{(i)}, p) \in C \times C$ such that $i \neq 0, p \notin B$, and $p^{(i)} \notin B$, then we have

$$\begin{pmatrix} h_{21} & \dots & h_{2t} \\ \multicolumn{3}{c}{\dots\dots\dots\dots} \\ h_{t1} & \dots & h_{tt} \end{pmatrix}_{X=p^{(i)}, Y=p} \begin{pmatrix} p_1^{(i)} - p_1 \\ \cdot \\ p_t^{(i)} - p_t \end{pmatrix} = \begin{pmatrix} 0 \\ \cdot \\ 0 \end{pmatrix}.$$

Since $p_1^{(i)} - p_1 = 0$, we have

$$H(p^{(i)}, p) \begin{pmatrix} p_2^{(i)} - p_2 \\ \cdot \\ p_t^{(i)} - p_t \end{pmatrix} = \begin{pmatrix} 0 \\ \cdot \\ 0 \end{pmatrix}.$$

Since $(p_2^{(i)} - p_2, \dots, p_t^{(i)} - p_t) \neq (0, \dots, 0)$, we have $\det H(p^{(i)}, p) = 0$. Since $p \notin B$ and $p^{(i)} \notin B$, we can take (x_1, y_1) as a local coordinate around $(p^{(i)}, p)$. Hence, from Lemma 4.1, there exists a holomorphic function $h(x_1, y_1)$ around $(p^{(i)}, p)$ such that $\det H(x, y) = (x_1 - y_1) h(x_1, y_1)$. Hence, $\Omega(x, y)$ is holomorphic at $(p^{(i)}, p)$.

□

Lemma 4.3. *Let $p \notin B$, s a local coordinate around p. Then, the expansion of $\Omega(x, y)$ in $s(y)$ at $s(x)$ is of the form*

$$\Omega(x, y) = \left(\frac{-1}{s(y) - s(x)} + regular \right) ds(x).$$

Proof. Set $Y = y$ in (6), then we have

$$F_i(X_1, \dots, X_t) = \sum_{j=1}^{t} h_{ij}(X, y) \cdot (X_j - y_j).$$

Hence, we obtain

$$\frac{\partial F_i}{\partial X_k}(x_1, \dots, x_t) = \sum_{j=1}^{t} \frac{\partial h_{ij}}{\partial X_k}(x, y) \cdot (x_j - y_j) + h_{ik}(x, y).$$

Set $x = y$, then we have

$$\frac{\partial F_i}{\partial X_k}(x_1, \dots, x_t) = h_{ik}(x, x).$$

Hence, we obtain $\det G_1(x) = \det H(x, x)$. On the other hand, since $p \notin B$, we can take (x_1, y_1) as a local coordinate around (p, p). Since $p \notin B$, we have $\det G_1(p) \neq 0$. In fact, if $\det G_1(p) = 0$, then $dx_1 / \det G_1(x)$ is not holomorphic at p, which contradicts Lemma 3.2 (ii). Hence, $\det H(x, y) / \det G_1(x)$ is holomorphic at (p, p). Hence, from Lemma 4.1, there exists a holomorphic function

$\tilde{h}(x_1, y_1)$ around (p,p) such that $\det H(x,y)/\det G_1(x) = 1 + (x_1 - y_1)\tilde{h}(x_1, y_1)$. Hence, we obtain Lemma 4.3.

□

Lemma 4.4. *When we express*

$$\det H(X,Y) = \sum \epsilon_{m_1,\ldots,m_t,n_1,\ldots,n_t} X_1^{m_1} \cdots X_t^{m_t} Y_1^{n_1} \cdots Y_t^{n_t},$$

we have $\sum_{k=1}^{t} a_k(m_k + n_k) \leq \sum_{k=2}^{t} a_k\left((d_{k-1}/d_k) - 1\right)$.

Proof. By the definition of h_{ij}, we have

$$h_{ij} = \frac{F_i(Y_1,\ldots,Y_{j-1},X_j,X_{j+1},\ldots,X_t) - F_i(Y_1,\ldots,Y_{j-1},Y_j,X_{j+1},\ldots,X_t)}{X_j - Y_j}.$$

When we express $F_i(X_1,\ldots,X_t) = \sum_{k=0}^{m} \tilde{F}_{ik}^{(j)}(X_1,\ldots,X_{j-1},X_{j+1},\ldots,X_t)X_j^k$, we have $h_{ij} = \sum_{k=0}^{m} \tilde{F}_{ik}^{(j)}(Y_1,\ldots,Y_{j-1},X_{j+1},\ldots,X_t)\sum_{l=0}^{k-1} X_j^l Y_j^{k-l-1}$. When we assign degrees as $\deg X_k = \deg Y_k = a_k$ and $\deg \lambda_{j_1,\ldots,j_t}^{(i)} = a_i d_{i-1}/d_i - \sum_{k=1}^{t} a_k j_k$, h_{ij} is a homogeneous polynomial of $\{\lambda_{j_1,\ldots,j_t}^{(i)}, X_k, Y_k\}$ of degree $a_i d_{i-1}/d_i - a_j$. Hence, we obtain Lemma 4.4.

□

Lemma 4.5. *The meromorphic bilinear form $d_y\Omega(x,y)$ is holomorphic except $\Delta \cup \{(p^{(i)},p) \mid i \neq 0,\ p \in B \text{ or } p^{(i)} \in B\} \cup C \times \{\infty\}$.*

Proof. It is sufficient to prove that $d_y\Omega(x,y)$ is holomorphic at (∞, y), $y \neq \infty$. From Lemma 4.4, with respect to x, we obtain

$$v_\infty\left(\det H(x,y)\right) \geq -\sum_{k=2}^{t} a_k\left((d_{k-1}/d_k) - 1\right).$$

If $v_\infty\left(\det H(x,y)\right) > -\sum_{k=2}^{t} a_k\left((d_{k-1}/d_k) - 1\right)$, then from Lemma 3.2 (ii) and Proposition 2.3 (iii) we obtain $v_\infty\left(\Omega(x,y)\right) \geq 0$. Hence, $d_y\Omega(x,y)$ is holomorphic at (∞, y). If $v_\infty\left(\det H(x,y)\right) = -\sum_{k=2}^{t} a_k\left((d_{k-1}/d_k) - 1\right)$, then $v_\infty\left(\Omega(x,y)\right) = -1$. Let s be a local coordinate around ∞, then from Lemma 4.4 there exists a constant e which does not depend on y such that

$$\Omega(x,y) = \left(\frac{e}{s} + \text{regular}\right) ds.$$

Hence, $d_y\Omega(x,y)$ is holomorphic at (∞, y), $y \neq \infty$.

□

Lemma 4.6. *Let ω be the normalized fundamental form. Then, there exist second kind defferentials $d\hat{r}_i$ $(1 \leq i \leq g)$ which are holomorphic except $\{\infty\}$ and satisfy the equation*

$$\omega(x,y) - d_y\Omega(x,y) = \sum_{i=1}^{g} du_i(x) d\hat{r}_i(y).$$

Proof. If we take $B_2 = \{(p^{(i)}, p) \mid p \in B\backslash\{\infty\} \text{ or } p^{(i)} \in B\backslash\{\infty\}\}$ in the proof of [15] Lemma 5, then the proof of Lemma 4.6 is similar to that of [15] Lemma 5.

$\square$

Lemma 4.7. *Let Q be the linear space consisting of meromorphic differentials on C which are singular only at ∞, and*

$$S := \left\{(x_1^{i_1} \cdots x_t^{i_t} / \det G_1(x))dx_1 \mid (i_1, \ldots, i_t) \in B(A_t)\right\}.$$

Then, S is a basis of Q.

Proof. For $\eta \in Q$, we consider the meromorphic function $\eta/\frac{dx_1}{\det G_1(x)}$. From Lemma 3.2 (ii), it may have a pole only at ∞. From Proposition 2.1 (i) and Proposition 2.2 (i), $\eta/\frac{dx_1}{\det G_1(x)}$ is a linear combination of $x_1^{i_1} \cdots x_t^{i_t}$ with $(i_1, \ldots, i_t) \in B(A_t)$, and the elements of S are linearly independent.

$\square$

Proof of Theorem 4.1 (i). Let us write

$$d_y\Omega(x, y) = \frac{\sum q_{i_1,\ldots,i_t,j_1,\ldots,j_t} x_1^{i_1} \cdots x_t^{i_t} y_1^{j_1} \cdots y_t^{i_t}}{(x_1 - y_1)^2 \det G_1(x) \det G_1(y)} dx_1 dy_1,$$

where $(i_1, \ldots, i_t), (j_1, \ldots, j_t) \in B(A_t)$, and $q_{i_1,\ldots,i_t,j_1,\ldots,j_t} \in \mathbb{C}$. Note that if $(m_1, \ldots, m_t) \in B(A_t)$, then $(m_1 + m, m_2, \ldots, m_t) \in B(A_t)$ for $m \in \mathbb{N}$. Hence, we obtain

$$\sum c_{i_1,\ldots,i_t,j_1,\ldots,j_t} \frac{x_1^{i_1} \cdots x_t^{i_t} y_1^{j_1} \cdots y_t^{j_t}}{\det G_1(x) \det G_1(y)}$$

$$= \frac{\sum (c_{i_1-2,\ldots,i_t,j_1,\ldots,j_t} - 2c_{i_1-1,\ldots,i_t,j_1-1,\ldots,j_t} + c_{i_1,\ldots,i_t,j_1-2,\ldots,j_t}) x_1^{i_1} \cdots x_t^{i_t} y_1^{j_1} \cdots y_t^{j_t}}{(x_1 - y_1)^2 \det G_1(x) \det G_1(y)},$$

where $(i_1, \ldots, i_t), (j_1, \ldots, j_t) \in B(A_t)$. Hence, $\hat{\omega}(x, y) = \hat{\omega}(y, x)$ is equivalent to

$$c_{i_1-2,\ldots,i_t,j_1,\ldots,j_t} - 2c_{i_1-1,\ldots,i_t,j_1-1,\ldots,j_t} + c_{i_1,\ldots,i_t,j_1-2,\ldots,j_t} - c_{j_1-2,\ldots,j_t,i_1,\ldots,i_t}$$

$$+2c_{j_1-1,\ldots,j_t,i_1-1,\ldots,i_t} - c_{j_1,\ldots,j_t,i_1-2,\ldots,i_t} = q_{j_1,\ldots,j_t,i_1,\ldots,i_t} - q_{i_1,\ldots,i_t,j_1,\ldots,j_t}.$$

By Lemma 4.6, 4.7, the system of the above linear equations has a solution.

$\square$

Proof of Theorem 4.1 (ii). From Lemma 4.6, $d_y\Omega(x, y)$ is holomorphic except $\Delta \cup C \times \{\infty\}$ and so is $\hat{\omega}$. Since $\hat{\omega}(x, y) = \hat{\omega}(y, x)$, $\hat{\omega}$ does not have a pole at $y = \infty$, and is holomorphic except Δ where it has a double pole. From the definition of dr_i, we obtain

$$\hat{\omega} - \omega = \sum_{i=1}^{g} du_i(x)(dr_i(y) - d\hat{r}_i(y)). \tag{7}$$

Both hand sides of (7) are meromorphic on $C \times C$. The singularities of the left hand side are contained in Δ, and those of right hand side are contained in $C \times \{\infty\}$. Hence, the possible singularity of $\hat{\omega} - \omega$ is $\{\infty\} \times \{\infty\}$. Hence, $\hat{\omega} - \omega$ and $dr_i - d\hat{r}_i$ are holomorphic on $C \times C$ and C respectively, which complete the proof of Theorem 4.1 (ii).

□

Proof of Theorem 4.1 (iii). The 1-form dr_i is a second kind differential. In fact $dr_i = d\hat{r}_i$ modulo holomorphic 1-form as is just proved in the proof of Theorem 4.1 (ii), and $d\hat{r}_i$ is a second kind differential by Lemma 4.6. Proof of Theorem 4.1 (iii) is similar to the case of the (n, s)-curves (cf. [15] Lemma 7,8, Proposition 3).

□

5 Sigma Functions for the Telescopic Curves

In this section, we construct the sigma function for a telescopic $(a_1, \dots, a_t)$-curve C. First, we take the following data.

1. A basis $\{\alpha_i, \beta_i\}_{i=1}^g$ of the homology group $H_1(C, \mathbb{Z})$ such that their intersection numbers are $\alpha_i \circ \alpha_j = \beta_i \circ \beta_j = 0$ and $\alpha_i \circ \beta_j = \delta_{ij}$.
2. The symplectic basis $\{du_i, dr_i\}_{i=1}^g$ of the first cohomology group $H^1(C, \mathbb{C})$ constructed in section 3 and 4.

We define the period matrices $\omega_1, \omega_2, \eta_1, \eta_2$ by

$$2\omega_1 = \left(\int_{\alpha_j} du_i \right),\ 2\omega_2 = \left(\int_{\beta_j} du_i \right),\ -2\eta_1 = \left(\int_{\alpha_j} dr_i \right),\ -2\eta_2 = \left(\int_{\beta_j} dr_i \right).$$

Then, ω_1 is invertible. Set $\tau = \omega_1^{-1}\omega_2$, then τ is symmetric and Im $\tau > 0$. By the Riemann's bilinear relation

$$2\pi i \eta \circ \eta' = \sum_{i=1}^{g} \left(\int_{\alpha_i} \eta \int_{\beta_i} \eta' - \int_{\alpha_i} \eta' \int_{\beta_i} \eta \right),$$

the matrix

$$M := \begin{pmatrix} \omega_1 & \omega_2 \\ \eta_1 & \eta_2 \end{pmatrix}$$

satisfies

$$M \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} {}^t M = -\frac{\pi\sqrt{-1}}{2} \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix},$$

where I_g denotes the unit matrix of degree g. Since $\eta_1 \omega_1^{-1}$ is symmetric (cf. [15] Lemma 8), we obtain the following proposition.

Proposition 5.1. (generalized Legendre relation)

$$ {}^t M \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} M = -\frac{\pi\sqrt{-1}}{2} \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}. $$

$\square$

Let $\delta = \tau\delta' + \delta''$ be the Riemann's constant of C with respect to our choice $(\infty, \{\alpha_i, \beta_i\}_{i=1}^g)$. Since the divisor of the holomorphic 1-form du_g is $(2g-2)\infty$, the Riemann's constant δ becomes a half period. Then, the sigma funtion $\sigma(u)$ associated with C is defined as follows.

DEFINITION 5.1. (Sigma function) For $u \in \mathbb{C}^g$,

$$ \begin{aligned} \sigma(u) = \sigma(u; M) &= c \cdot \exp\left(\frac{1}{2}\, {}^t u \eta_1 \omega_1^{-1} u\right) \theta[\delta]\ ((2\omega_1)^{-1}u, \tau) \\ &= c \cdot \exp\left(\frac{1}{2}\, {}^t u \eta_1 \omega_1^{-1} u\right) \\ \times \sum_{n \in \mathbb{Z}^g} &\exp\left\{\pi\sqrt{-1}\, {}^t(n+\delta')\tau(n+\delta') + 2\pi\sqrt{-1}\, {}^t(n+\delta')((2\omega_1)^{-1}u + \delta'')\right\}, \end{aligned} $$

where c is a constant.

By Proposition 5.1, we obtain the following proposition.

Proposition 5.2. For any $m_1, m_2 \in \mathbb{Z}^g$, and $u \in \mathbb{C}^g$, we have

$$ \begin{aligned} \sigma(u + 2\omega_1 m_1 + 2\omega_2 m_2)/\sigma(u) &= \exp\left(\pi\sqrt{-1}\left({}^t m_1 m_2 + 2\,{}^t\delta' m_1 - 2\,{}^t\delta'' m_2\right)\right) \\ &\times \exp\left({}^t(2\eta_1 m_1 + 2\eta_2 m_2)(u + \omega_1 m_1 + \omega_2 m_2)\right). \end{aligned} $$

$\square$

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Appendix

A Proof of Proposition 2.1

Lemma A.1. $V(A_t) + \mathbb{N}^t = \mathbb{N}^t \backslash B(A_t)$.

Proof. Since $M + N \notin B(A_t)$ for $M \notin B(A_t)$ and $N \in \mathbb{N}^t$, we have $V(A_t) + \mathbb{N}^t \subset \mathbb{N}^t \backslash B(A_t)$. Suppose $V(A_t) + \mathbb{N}^t \subsetneq \mathbb{N}^t \backslash B(A_t)$. Take $M_1 \in \mathbb{N}^t \backslash B(A_t)$ such that $M_1 \notin V(A_t) + \mathbb{N}^t$. Since $M_1 \notin V(A_t)$ and $M_1 \notin B(A_t)$, there exist $M_2 \in \mathbb{N}^t \backslash B(A_t)$ and $(0, \ldots, 0) \neq N_1 \in \mathbb{N}^t$ such that $M_1 = M_2 + N_1$. Since $M_1 \notin V(A_t) + \mathbb{N}^t$, we have $M_2 \notin V(A_t) + \mathbb{N}^t$. Similarly, for $M_i \in \mathbb{N}^t \backslash B(A_t)$ and $M_i \notin V(A_t) + \mathbb{N}^t$, we have $M_i = M_{i+1} + N_i$ with $M_{i+1} \in \mathbb{N}^t \backslash B(A_t)$, $M_{i+1} \notin V(A_t) + \mathbb{N}^t$, and $(0, \ldots, 0) \neq N_i \in \mathbb{N}^t$. Hence, there exists a infinite sequence such that $\Psi(M_1) > \Psi(M_2) > \cdots > \Psi(M_i) > \cdots$. This is contradiction.

$\square$

Proof of Proposition 2.1 (i). From (2), it is sufficient to prove

$$\mathrm{Span}\{X^N \mid N \in B(A_t)\} + (\{F_M \mid M \in V(A_t)\}) = \mathbb{C}[X].$$

We prove that for any $T \in \mathbb{N}^t$

$$X^T \in \mathrm{Span}\{X^N \mid N \in B(A_t)\} + (\{F_M \mid M \in V(A_t)\}),$$

by transfinite induction with respect to the well-order $<$ in $\mathbb{N}^t$. The statement is correct for the minimal element $T = (0, \ldots, 0)$. Suppose that it is correct for any $U \in \mathbb{N}^t$ such that $U < T$. Since it is correct for $T \in B(A_t)$, we assume $T \notin B(A_t)$. From Lemma A.1, there exist $M \in V(A_t)$ and $Z \in \mathbb{N}^t$ such that $T = M + Z$. Then, we have $X^T = X^M X^Z = (X^M - F_M)X^Z + F_M X^Z$, where $U < T$ for any term X^U appearing in $(X^M - F_M)X^Z$. Hence, by the assumption of transfinite induction, we obtain the statement.

$\square$

We define the function $o : R \to \mathbb{N} \cup \{-\infty\}$ by

$$o(f) := \begin{cases} -\infty & \text{for } f = 0 \\ \max\{\Psi(N) \mid \lambda_N \neq 0\} & \text{for } f \neq 0 \end{cases},$$

where for $f \neq 0$ we express $f = \sum_N \lambda_N x^N$ with $\lambda_N \in \mathbb{C}$ and $N \in B(A_t)$.

Lemma A.2. *$o(x^T) = \Psi(T)$ for any $T \in \mathbb{N}^t$.*

Proof. We prove the statement by transfinite induction with respect to the well-order $<$ in $\mathbb{N}^t$. It is correct for the minimal element $T = (0, \ldots, 0) \in \mathbb{N}^t$. Suppose that it is correct for any $U \in \mathbb{N}^t$ such that $U < T$. Since it is correct for $T \in B(A_t)$, we assume $T \notin B(A_t)$. From Lemma A.1, there exist $M \in V(A_t)$ and $Z \in \mathbb{N}^t$ such that $T = M + Z$. Then, we have $X^T = X^M X^Z = (X^M - F_M)X^Z + F_M X^Z$. Since $X^M - F_M = X^L + \sum_N \lambda_N X^N$ from (1), we have $x^T = (x^L + \sum_N \lambda_N x^N)x^Z = x^{L+Z} + \sum_N \lambda_N x^{N+Z}$. Since $N + Z < L + Z < T$, by the assumption of transfinite induction, we have $o(x^{L+Z}) = \Psi(L + Z)$ and $o(x^{N+Z}) = \Psi(N+Z)$. Since $o(f+g) = \max\{o(f), o(g)\}$ for $f, g \in R$ with $o(f) \neq o(g)$, we have $o(x^T) = o(x^{L+Z} + \sum_N \lambda_N x^{N+Z}) = o(x^{L+Z}) = \Psi(L+Z) = \Psi(T)$.

□

Lemma A.3. *The function o satisfies the following conditions:*

(i) $o(f) = -\infty$ *if and only if* $f = 0$,

(ii) $o(fg) = o(f) + o(g)$ *for any* $f, g \in R$, *where we define* $-\infty + (-\infty) = a + (-\infty) = (-\infty) + a = -\infty$ *for* $a \in \mathbb{N}$,

(iii) $o(f + g) \leq \max\{o(f), o(g)\}$,

(iv) $o(R \backslash \{0\}) = \langle A_t \rangle$, *in particular* $\mathbb{N} \backslash o(R \backslash \{0\})$ *is a finite set, and*

(v) $o(a) = 0$ *for any* $0 \neq a \in \mathbb{C}$.

Proof. (i), (iii), (v), and $o(R\backslash\{0\}) = \langle A_t \rangle$ are trivial. Since $\mathrm{GCD}\{a_1, \ldots, a_t\} = 1$, $\mathbb{N}\backslash\langle A_t \rangle$ is a finite set (cf. [16], Theorem 5). We prove (ii). If $f = 0$ or $g = 0$, then $o(fg) = o(f) + o(g) = -\infty$. Suppose $f \neq 0$ and $g \neq 0$. Then, we can express

$$f = \lambda_M x^M + \sum_T \lambda_T x^T \text{ and } g = \tilde{\lambda}_N x^N + \sum_Z \tilde{\lambda}_Z x^Z,$$

where $\lambda_M, \lambda_T, \tilde{\lambda}_N, \tilde{\lambda}_Z \in \mathbb{C}$, $\lambda_M \neq 0, \tilde{\lambda}_N \neq 0$, $M, T, N, Z \in B(A_t)$, $\Psi(T) < \Psi(M)$, and $\Psi(Z) < \Psi(N)$. From Lemma A.2, we have $o(fg) = o(\lambda_M \tilde{\lambda}_N x^{M+N}) = \Psi(M + N) = \Psi(M) + \Psi(N) = o(f) + o(g)$.

□

Proof of Proposition (ii). Take $f, g \in R$ such that $fg = 0$. Then, since $-\infty = o(fg) = o(f) + o(g)$, we have $o(f) = -\infty$ or $o(g) = -\infty$. Hence, we obtain $f = 0$ or $g = 0$.

□

Lemma A.4. *Let* $B \subset \mathbb{N}^t$ *be a set such that the restriction map of* $\Psi : \mathbb{N}^t \to \langle A_t \rangle$ *on* B *is bijective. Then, the set* $\{x^M \mid M \in B\} \subset R$ *is a basis of* R *over* $\mathbb{C}$.

Proof. Since $o(x^T) = \Psi(T)$ for $T \in \mathbb{N}^t$ and $o(f + g) = \max\{o(f), o(g)\}$ for $f, g \in R$ with $o(f) \neq o(g)$, the set $\{x^M \mid M \in B\}$ is linearly independent. Since $R = \mathrm{Span}\{x^N \mid N \in B(A_t)\}$, in order to prove $R = \mathrm{Span}\{x^M \mid M \in B\}$, it is sufficient to prove $\mathrm{Span}\{x^N \mid N \in B(A_t)\} \subset \mathrm{Span}\{x^M \mid M \in B\}$. We prove $\mathrm{Span}\{x^N \mid N \in B(A_t),\ \Psi(N) \leq m\} \subset \mathrm{Span}\{x^M \mid M \in B,\ \Psi(M) \leq m\}$ for any $m \in \mathbb{N}$ by induction. For $m = 0$, the statement is trivial. Suppose that the statement is correct for any i with $0 \leq i \leq m - 1$. If $m \notin \langle A_t \rangle$, then since $\mathrm{Span}\{x^M \mid M \in B,\ \Psi(M) \leq m\} = \mathrm{Span}\{x^M \mid M \in B,\ \Psi(M) \leq m - 1\}$ and $\mathrm{Span}\{x^N \mid N \in B(A_t),\ \Psi(N) \leq m\} = \mathrm{Span}\{x^N \mid N \in B(A_t),\ \Psi(N) \leq m - 1\}$, the statement is correct. Suppose $m \in \langle A_t \rangle$. Take $T \in B$ such that $\Psi(T) = m$. If $T \in B(A_t)$, then since $\mathrm{Span}\{x^M \mid M \in B,\ \Psi(M) \leq m\} = \mathrm{Span}\{x^M \mid M \in B,\ \Psi(M) \leq m - 1\} \cup \mathbb{C}\{x^T\}$ and $\mathrm{Span}\{x^N \mid N \in B(A_t),\ \Psi(N) \leq m\} = \mathrm{Span}\{x^N \mid N \in B(A_t),\ \Psi(N) \leq m - 1\} \cup \mathbb{C}\{x^T\}$, the statement is correct. Suppose $T \notin B(A_t)$. Then, we can express $x^T = \lambda_L x^L + \sum_N \lambda_N x^N$, where $0 \neq \lambda_L, \lambda_N \in \mathbb{C}$, $L, N \in B(A_t)$, $\Psi(L) = m$, and $\Psi(N) \leq m - 1$. Since

$x^L = \lambda_L^{-1}(x^T - \sum_N \lambda_N x^N) \in \mathrm{Span}\{x^N \mid N \in B(A_t),\ \Psi(N) \leq m-1\} \cup \mathbb{C}\{x^T\} \subset \mathrm{Span}\{x^M \mid M \in B,\ \Psi(M) \leq m-1\} \cup \mathbb{C}\{x^T\} \subset \mathrm{Span}\{x^M \mid M \in B,\ \Psi(M) \leq m\}$, we have $\mathrm{Span}\{x^N \mid N \in B(A_t),\ \Psi(N) \leq m\} \subset \mathrm{Span}\{x^M \mid M \in B,\ \Psi(M) \leq m\}$.

□

Lemma A.5. *Given i, there exists a set $T_i \subset \mathbb{N}^{i-1} \times \{0\} \times \mathbb{N}^{t-i}$ such that $\sharp T_i = a_i$ and the restriction map of $\Psi : \mathbb{N}^t \to \langle A_t \rangle$ on B_i is bijective, where $B_i := T_i + \{0\}^{i-1} \times \mathbb{N} \times \{0\}^{t-i}$.*

Proof. Since $\mathrm{GCD}\{a_1, \dots, a_t\} = 1$, the set $\{c \in a_1\mathbb{N} + \cdots + a_{i-1}\mathbb{N} + a_{i+1}\mathbb{N} + \cdots + a_t\mathbb{N} \mid c \equiv\ j \bmod a_i\}$ is not empty for any j with $0 \leq j \leq a_i - 1$. Let $c_j := \min\{c \in a_1\mathbb{N} + \cdots + a_{i-1}\mathbb{N} + a_{i+1}\mathbb{N} + \cdots + a_t\mathbb{N} \mid c \equiv\ j \bmod a_i\}$, $N_j \in \mathbb{N}^{i-1} \times \{0\} \times \mathbb{N}^{t-i}$ such that $\Psi(N_j) = c_j$, and $T_i := \{N_j \mid 0 \leq j \leq a_i - 1\}$, then T_i satisfies the condition of Lemma A.5.

□

Proof of Proposition 2.1 (iii). Since $o(x^T) = \Psi(T)$ for $T \in \mathbb{N}^t$ and $o(f + g) = \max\{o(f), o(g)\}$ for $f, g \in R$ with $o(f) \neq o(g)$, the set $\{x^M \mid M \in \{0\}^{i-1} \times \mathbb{N} \times \{0\}^{t-i}\} \subset \mathbb{C}[x_i]$ is linearly independent. Hence, the extension of field $\mathbb{C}(x_i)/\mathbb{C}$ is a simple transcendental extension for any i. Next, we prove $[K : \mathbb{C}(x_i)] \leq a_i$ for any i. From Lemma A.4 and Lemma A.5, we have $R = \mathbb{C}[x_1, \dots, x_t] = \mathrm{Span}\{x^M \mid M \in T_i + \{0\}^{i-1} \times \mathbb{N} \times \{0\}^{t-i}\}$. Hence, $\mathbb{C}[x_1, \dots, x_t] = \mathbb{C}[x_i]f_0 + \cdots + \mathbb{C}[x_i]f_{a_i-1}$, where $f_j := x^{N_j}$ (see the proof of Lemma A.5 for N_j). Since $f_0 = 1$, we obtain the finite extension of integral domain $\mathbb{C}(x_i) \subset \mathbb{C}(x_i)f_0 + \cdots + \mathbb{C}(x_i)f_{a_i-1}$. Since $\mathbb{C}(x_i)$ is a field, $\mathbb{C}(x_i)f_0 + \cdots + \mathbb{C}(x_i)f_{a_i-1}$ is also a field. Hence, we obtain $\mathbb{C}(x_i) \subset \mathbb{C}(x_i)f_0 + \cdots + \mathbb{C}(x_i)f_{a_i-1} = K$, and $[K : \mathbb{C}(x_i)] \leq a_i$.

□

Proof of Proposition 2.1 (iv). We define the function $v_\infty : K \to \mathbb{Z} \cup \{\infty\}$ by

$$v_\infty(f) := \begin{cases} \infty & \text{for } f = 0 \\ -o(f_1) + o(f_2) & \text{for } f \neq 0 \end{cases},$$

where for $f \neq 0$ we express $f = f_1/f_2$ with $f_1, f_2 \in R$. The definition of v_∞ is well-defined. In fact, if $0 \neq f = f_1/f_2 = g_1/g_2$, then since $f_1 g_2 = g_1 f_2 \in R$, we have $o(f_1) + o(g_2) = o(f_1 g_2) = o(g_1 f_2) = o(g_1) + o(f_2)$. From Lemma A.3, one can check that the function v_∞ is a discrete valuation of K. From Lemma A.2, we obtain $v_\infty(x_i) = -a_i$. From [19] p.19 Theorem 1.4.11, we obtain $[K : \mathbb{C}(x_i)] = \deg(x_i)_\infty \geq \deg(a_i v_\infty) = a_i$. On the other hand, in the proof of Proposition 2.1 (iii), we proved $[K : \mathbb{C}(x_i)] \leq a_i$. Hence, we obtain $(x_i)_\infty = a_i v_\infty$.

□

B Proof of Proposition 2.2

Proof of Proposition 2.2 (i). It is trivial that $R \subset \bigcup_{k=0}^{\infty} L(kv_\infty)$. On the other hand, we have

$$\bigcup_{k=0}^{\infty} L(kv_\infty) \subset \bigcap_{v \neq v_\infty} \mathcal{O}_v \subset \bigcap_{p \in C^{\mathrm{aff}}} \mathcal{O}_p = R,$$

where $\mathcal{O}_v := \{f \in K \mid v(f) \geq 0\}$ and $\mathcal{O}_p := \{f \in K \mid v_p(f) \geq 0\}$ (see Proposition 2.2 (ii) for v_p).

□

Proof of Proposition 2.2 (ii). It is trivial that the map ϕ is injective. We prove that the map ϕ is surjective. Let v be a discrete valuation such that $v \neq v_\infty$. Since $v(x_i) \geq 0$ for any i, we have $R \subset \mathcal{O}_v$. Let P be the maximal ideal of $\mathcal{O}_v$ and $m := P \cap R$, then we have

$$\mathbb{C} \hookrightarrow R/m \hookrightarrow \mathcal{O}_v/P.$$

Since $[\mathcal{O}_v/P : \mathbb{C}] = 1$, we have $\mathbb{C} \simeq R/m \simeq \mathcal{O}_v/P$. Hence, m is a maximal ideal. Let R_m be the localization of R with respect to m, then R_m and $\mathcal{O}_v$ are discrete valuation rings with $R_m \subset \mathcal{O}_v$ and $P \cap R_m = mR_m$. Hence, from [8] p.40 Theorem 6.1A, we obtain $R_m = \mathcal{O}_v$. Since there exists $p \in C^{\mathrm{aff}}$ such that $\mathcal{O}_p = R_m$, we have $\mathcal{O}_p = \mathcal{O}_v$. Hence, we obtain $v_p = v$, and the map ϕ is surjective.

□

C Proof of Proposition 2.3

Let $T(A_t) := B(A_t) \cap \left(\{0\} \times \mathbb{N}^{t-1}\right)$.

Lemma C.1. *(i)* $T(A_t) = \{M(b_i) \in B(A_t) \mid i = 0, \dots, a_1 - 1\}$*, where* $b_i := \min\{b \in a_2\mathbb{N} + \cdots + a_t\mathbb{N} \mid b \equiv i \ mod \ a_1\}$ *for* $i = 0, \dots, a_1 - 1$*. In particular,* $\sharp T(A_t) = a_1$.

(ii) $B(A_t) = T(A_t) + \mathbb{N} \times \{0\}^{t-1}$.

(iii) $V(A_t) \subset \{T(A_t) + \{(0, \dots, 0, 1, 0, \dots, 0) \mid i = 2, \dots, t\}\} \backslash T(A_t) \subset \{0\} \times \mathbb{N}^{t-1}$*, where the* i*-th component of* $(0, \dots, 0, 1, 0, \dots, 0)$ *is equal to 1.*

(iv) The set $\{0\}^{i-1} \times \mathbb{N} \times \{0\}^{t-i} \cap V(A_t)$ *consists of only one element for any* i $(2 \leq i \leq t)$.

Proof. We have $M(b_i) := (m_1, \dots, m_t) \in \{0\} \times \mathbb{N}^{t-1}$. In fact, if $m_1 \neq 0$, then we have $\Psi((0, m_2, \dots, m_t)) \equiv b_i \equiv i \bmod a_1$ and $\Psi((0, m_2, \dots, m_t)) < b_i$, which contradicts the definition of b_i. Hence, we have $M(b_i) \in T(A_t)$. For $M, N \in \{0\} \times \mathbb{N}^{t-1}$ such that $\Psi(M) > \Psi(N)$ and $\Psi(M) - \Psi(N) = ea_1$ for some $e \in \mathbb{N}_+$, we have $M \notin T(A_t)$. In fact, for $N' := (e, 0, \dots, 0) + N$, we have $M > N'$ and $\Psi(M) = \Psi(N')$, which means $M \notin B(A_t)$. Hence, we obtain (i).

Next, we prove $B(A_t) \subset T(A_t) + \mathbb{N} \times \{0\}^{t-1}$. Let $M := (m_1, \ldots, m_t) \in B(A_t)$, $M_1 := (0, m_2, \ldots, m_t)$, and $M_2 := (m_1, 0, \ldots, 0)$. Since $M_1 + M_2 \in B(A_t)$, we have $M_1, M_2 \in B(A_t)$. Since $M_1 \in B(A_t) \cap (\{0\} \times \mathbb{N}^{t-1}) = T(A_t)$, we have $M \in T(A_t) + \mathbb{N} \times \{0\}^{t-1}$. Suppose $B(A_t) \subsetneq T(A_t) + \mathbb{N} \times \{0\}^{t-1}$. Then, from (i), there exist i $(0 \leq i \leq a_1 - 1)$ and $M_3 \in \mathbb{N} \times \{0\}^{t-1}$ such that $M(b_i) + M_3 \notin B(A_t)$. Take $N \in B(A_t)$ such that $\Psi(M(b_i) + M_3) = \Psi(N)$. Since $N \in B(A_t) \subset T(A_t) + \mathbb{N} \times \{0\}^{t-1}$ and $\Psi(N) \equiv i \bmod a_1$, there exists $M_4 \in \mathbb{N} \times \{0\}^{t-1}$ such that $N = M(b_i) + M_4$. Hence, $M_3 > M_4$, $M_3, M_4 \in \mathbb{N} \times \{0\}^{t-1}$, and $\Psi(M_3) = \Psi(M_4)$, which is contradiction. Hence, we obtain $B(A_t) = T(A_t) + \mathbb{N} \times \{0\}^{t-1}$.

Next, we prove $V(A_t) \subset \{0\} \times \mathbb{N}^{t-1}$. Let $M := (m_1, \ldots, m_t) \in V(A_t)$, $M_1 := (0, m_2, \ldots, m_t)$, and $M_2 := (m_1, 0, \ldots, 0)$. Since $M \notin B(A_t)$ and $M_2 \in B(A_t)$, we have $M_1 \notin B(A_t)$. From the definition of $V(A_t)$, we obtain $M_2 = (0, \ldots, 0)$. Hence, we obtain $V(A_t) \subset \{0\} \times \mathbb{N}^{t-1}$. Let $M \in V(A_t) \subset \{0\} \times \mathbb{N}^{t-1}$. Since $M \neq (0, \ldots, 0)$, there exist i $(2 \leq i \leq t)$ and $M_1 \in \{0\} \times \mathbb{N}^{t-1}$ such that $M = M_1 + (0, \ldots, 0, 1, 0, \ldots, 0)$, where the i-th component of $(0, \ldots, 0, 1, 0, \ldots, 0)$ is equal to 1. Since $M_1 \in B(A_t)$ from the definition of $V(A_t)$, we have $M_1 \in B(A_t) \cap \left(\{0\} \times \mathbb{N}^{t-1}\right) = T(A_t)$. Hence, we obtain (iii).

For $2 \leq i \leq t$, the set $\{0\}^{i-1} \times \mathbb{N} \times \{0\}^{t-i} \cap \{\mathbb{N}^t \backslash B(A_t)\}$ is not empty. In fact, since

$$\Psi((0, \ldots, 0, a_1, 0, \ldots, 0)) = \Psi((a_i, 0, \ldots, 0)) = a_1 a_i,$$

we have $(0, \ldots, 0, a_1, 0, \ldots, 0) > (a_i, 0, \ldots, 0)$. Let N_i be the minimal element of $\{0\}^{i-1} \times \mathbb{N} \times \{0\}^{t-i} \cap \{\mathbb{N}^t \backslash B(A_t)\}$. Then, we obtain $\{0\}^{i-1} \times \mathbb{N} \times \{0\}^{t-i} \cap V(A_t) = \{N_i\}$. Hence, we obtain (iv).

□

Let $SV(A_t) = \{N_i \mid 2 \leq i \leq t\}$ (see the proof of Lemma C.1 (iv) for N_i). For $F := \sum \lambda_N X^N \in \mathbb{C}[X]$, we define *multideg* of F by

$$\mathrm{multideg}(F) := \begin{cases} -\infty & \text{for } F = 0 \\ \max\{N \in \mathbb{N}^t \mid \lambda_N \neq 0\} & \text{for } F \neq 0 \end{cases}.$$

Also, we define *leading term* of F by

$$\mathrm{LT}(F) := \begin{cases} 0 & \text{for } F = 0 \\ \lambda_T X^T & \text{for } F \neq 0, \text{ where } T = \mathrm{multideg}(F) \end{cases}.$$

For a ideal $J \subset \mathbb{C}[X]$, we define

$$\Delta(J) := \mathbb{N}^t \backslash \bigcup_{F \in J \backslash \{0\}} \{\mathrm{multideg}(F) + \mathbb{N}^t\}.$$

Then, we have

$$\mathrm{Span}\{X^M \mid M \in \Delta(J)\} \cap J = \{0\}. \tag{8}$$

Lemma C.2. *(i)* $\{F_M \mid M \in SV(A_t)\}$ *is a Gröbner basis of the ideal* $J := (\{F_M \mid M \in SV(A_t)\})$ *with respect to the order* $<$ *in* $\mathbb{N}^t$*, i.e.,* $(\{LT(F) \mid F \in J\}) = (\{LT(F_M) \mid M \in SV(A_t)\})$.

(ii) $Span\{X^N \mid N \in B(A_t)\} \cap (\{F_M \mid M \in SV(A_t)\}) = \{0\}$.

Proof. For $M, N \in SV(A_t)$ with $M \neq N$, we have L.C.M.$\{\mathrm{LT}(F_M), \mathrm{LT}(F_N)\} = \mathrm{LT}(F_M)\mathrm{LT}(F_N)$. Hence, from [6] p.102 Theorem 3 and p.103 Proposition 4, we obtain (i). From (i), we obtain $\Delta\left((\{F_M \mid M \in SV(A_t)\})\right) = \mathbb{N}^t \backslash \{SV(A_t) + \mathbb{N}^t\} \supset \mathbb{N}^t \backslash \{V(A_t) + \mathbb{N}^t\} = B(A_t)$, where the last equality is due to Lemma A.1. Since Span$\{X^N \mid N \in \Delta(\{F_M \mid M \in SV(A_t)\})\} \cap (\{F_M \mid M \in SV(A_t)\}) = \{0\}$ from (8), we have Span$\{X^N \mid N \in B(A_t)\} \cap (\{F_M \mid M \in SV(A_t)\}) = \{0\}$.

□

Lemma C.3. *If A_t is telescopic, then the followings are satisfied.*

(i) $T(A_t) = \{(0, m_2, \ldots, m_t) \in \mathbb{N}^t \mid 0 \leq m_i \leq d_{i-1}/d_i - 1, i = 2, \ldots, t\}$.

(ii) $SV(A_t) = V(A_t) = \{(0, \ldots, 0, d_{i-1}/d_i, 0, \ldots, 0) \in \{0\}^{i-1} \times \mathbb{N} \times \{0\}^{t-i} \mid 2 \leq i \leq t\}$.

Proof. Let $U := \{(0, m_2, \ldots, m_t) \in \mathbb{N}^t \mid 0 \leq m_i \leq d_{i-1}/d_i - 1, i = 2, \ldots, t\}$, $u := (0, u_2, \ldots, u_t) \in U$, and $v := (0, v_2, \ldots, v_t) \in U$ with $u \neq v$. First, we prove $\Psi(u) \not\equiv \Psi(v) \bmod a_1$. Suppose that there exists an integer w such that $\Psi(u) - \Psi(v) = wa_1$. Let ρ be the positive integer such that $u_\rho \neq v_\rho$, $u_{\rho+1} = v_{\rho+1}, \ldots, u_t = v_t$. Without loss of generality, we assume $u_\rho > v_\rho$. Then, we have $(u_\rho - v_\rho)a_\rho = wa_1 - \sum_{k=2}^{\rho-1}(u_k - v_k)a_k$ and $0 < u_\rho - v_\rho < d_{\rho-1}/d_\rho$, which is contradiction. Hence, we obtain $\Psi(u) \not\equiv \Psi(v) \bmod a_1$. Since A_t is telescopic, for any $u = (0, u_2, \ldots, u_t) \in \mathbb{N}^t$, there exists $u' \in U$ such that $\Psi(u) \equiv \Psi(u') \bmod a_1$. Since $\Psi(u) \geq \Psi(u')$ and $\sharp U = a_1$, we have $\{\Psi(u) \mid u \in U\} = \{b_0, \ldots, b_{a_1-1}\}$, where $b_i := \min\{b \in a_2\mathbb{N} + \cdots + a_t\mathbb{N} \mid b \equiv i \bmod a_1\}$. Finally, we prove $u \in B(A_t)$ for any $u \in U$. Take $u \in U$, then there exists $u'' = (u''_1, \ldots, u''_t) \in B(A_t)$ such that $\Psi(u) = \Psi(u'')$. Since A_t is telescopic, we have $0 \leq u''_j < d_{j-1}/d_j$ for $2 \leq j \leq t$. Since $u''_1 = 0$ from the definition of b_i, we obtain $u'' \in U$. Hence, we obtain $u = u'' \in B(A_t)$. From Lemma C.1 (i), we obtain (i). From Lemma C.1 (iii) (iv) and the definition of $V(A_t)$, we obtain (ii).

□

Proof of Proposition 2.3. From Lemma C.2 (ii) and Lemma C.3 (ii), the condition (2) is met. From Lemma C.1 (ii) and Lemma C.3 (i), we obtain Proposition 2.3 (i). From Lemma C.3 (ii), we obtain Proposition 2.3 (ii). From Proposition 2.1 (i) and Proposition 2.2 (i), the gap values at ∞ are $\mathbb{N} \backslash \langle A_t \rangle$. Hence, from [16] Theorem 5, we obtain Proposition 2.3 (iii).

□

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